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The stability of elastico-viscous flow between rotating cylinders

Part 3. Overstability in viscous and Maxwell fluids

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Consideration is given to the possibility of overstability in the Couette flow of viscous and elastico-viscous liquids. The relevant linear perturbation equations are solved numerically using an initial-value technique. It is shown that overstability is not possible in the case of Newtonian liquids for the cases considered. In contrast, overstability is to be expected in the case of moderately-elastic Maxwell liquids. The Taylor number associated with the overstable mode decreases steadily as the amount of elasticity in the liquid increases, and it is concluded that highly elastic Maxwell liquids can be very unstable indeed.

Introduction

In the first two papers under the same title (Thomas & Walters 1964*a*, *b*) it was shown that the presence of elasticity in an elastico-viscous liquid (and in particular the 'Maxwell' liquid) can be a significant destabilizing agent. Evidence was also given which suggested that the principle of exchange of stabilities does not hold for highly elastic liquids. In the present paper, this latter problem is considered in some detail for a Maxwell liquid. It is shown that overstability sets in much earlier (i.e. at a lower value of an elastic parameter k) than one would have concluded from the earlier work. In fact, over quite a wide range of the elastic parameter, two modes of instability are possible—the usual 'stationary' stability mode and an 'overstable' mode with a lower Taylor number.

The problem of overstability in the Couette flow of Newtonian fluids has often been posed (see, for example, Chandrasekhar 1961), but so far as the authors are aware, no complete solution has appeared in the literature. This particular problem is a special case of the one discussed in the present paper, and is considered in the discussion of the results in § 4.

2. The basic equations

The particular elastico-viscous liquid considered in the earlier papers was that designated liquid B' by Walters (1964) with equations of state†

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{1}$$

$$p'^{ik}(x,t) = 2\int_{-\infty}^{t} \Psi(t-t')\frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^r}e^{(1)mr}(x',t')\,dt', \tag{2}$$

where p_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor of a fixed co-ordinate system x^i, $e^{(1)}_{ik}$ the rate-of-strain tensor, and

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau}e^{-(t-t')/\tau}\,d\tau. \tag{3}$$

In these equations, $N(\tau)$ is the distribution function of relaxation times τ (Walters 1960) and $x'^i (= x'^i(x,t,t'))$ is the position at time t' of the element that is instantaneously at the point x^i at time t. The Newtonian liquid of constant viscosity η_0 is a special case of liquid B', obtained by writing‡

$$N(\tau) = \eta_0\,\delta(\tau) \tag{4}$$

in equations (2) and (3).

In the present problem, we shall consider the x^i co-ordinate system to be a cylindrical polar system (r,θ,z), the axis of the cylinders being along the z axis and the inner and outer cylinders having radii r_1 and r_2, respectively. We shall further assume that the inner and outer cylinders rotate about their common axis with angular velocities Ω_1 and Ω_2, respectively.

The 'steady-state' velocity distribution is given by (Thomas & Walters 1964*a*)

$$v_{(r)} = 0, \quad v_{(\theta)} = V(r), \quad v_{(z)} = 0, \tag{5}$$

where
$$V = Cr + D/r, \tag{6}$$

and
$$C = \frac{r_2^2\,\Omega_2 - r_1^2\,\Omega_1}{r_2^2 - r_1^2}, \quad D = \frac{r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r_2^2 - r_1^2}, \tag{7}$$

$v_{(r)}, v_{(\theta)}, v_{(z)}$ being the physical components of the velocity vector.

We shall consider the behaviour of liquid B' when the steady state described by (5)–(7) is disturbed slightly, assuming a velocity distribution of the form §

$$v_{(r)} = u\,e^{i\sigma t}, \quad v_{(\theta)} = V + v\,e^{i\sigma t}, \quad v_{(z)} = w\,e^{i\sigma t}, \tag{8}$$

† Covariant suffices are written below, contravariant suffices above, and the usual summation convention for repeated suffices is assumed.

‡ δ denotes a Dirac delta-function, defined in such a way that

$$\delta(x) = 0 \quad (x \neq 0), \quad \int_{-\infty}^{\infty}\delta(x)\,dx = \int_0^\infty \delta(x)\,dx = 1.$$

§ In equation (8) and in subsequent equations, the real part is to be understood.

where u, v and w are complex functions of r and z, whose powers higher than the first can be neglected.† In parts 1 and 2, consideration was confined to the case of neutral stability. It was also assumed that the principle of exchange of stabilities was valid, so that σ could be set equal to zero in equation (8). In the present paper, only neutral stability is assumed, which restricts σ to be real.

In order to determine the stress components corresponding to the velocity distribution (8), it is necessary to determine the displacement functions x'^i, which we shall write as r', θ', z'. These are given by (cf. Oldroyd 1950, equation (21))

$$\left.\begin{aligned} \frac{\partial r'}{\partial t}+v_{(r)}\frac{\partial r'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial r'}{\partial \theta}+v_{(z)}\frac{\partial r'}{\partial z} &= 0,\\ \frac{\partial \theta'}{\partial t}+v_{(r)}\frac{\partial \theta'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial \theta'}{\partial \theta}+v_{(z)}\frac{\partial \theta'}{\partial z} &= 0,\\ \frac{\partial z'}{\partial t}+v_{(r)}\frac{\partial z'}{\partial r}+\frac{v_{(\theta)}}{r}\frac{\partial z'}{\partial \theta}+v_{(z)}\frac{\partial z'}{\partial z} &= 0. \end{aligned}\right\} \tag{9}$$

Solving these equations for r', θ', z', we obtain

$$\left.\begin{aligned} r' &= r-u(e^{i\sigma t}-e^{i\sigma t'})/i\sigma,\\ \theta' &= \theta-(t-t')\frac{V}{r}+\frac{u}{i\sigma}\frac{d}{dr}\left(\frac{V}{r}\right)\left[(t-t')\,e^{i\sigma t}-\frac{1}{i\sigma}(e^{i\sigma t}-e^{i\sigma t'})\right]-\frac{v}{i\sigma r}(e^{i\sigma t}-e^{i\sigma t'}),\\ z' &= z-w(e^{i\sigma t}-e^{i\sigma t'})/i\sigma. \end{aligned}\right\} \tag{10}$$

In order to obtain the rate-of-strain components $e^{(1)mr}(x',t')$ that appear in the equations of state (2), we first write down the rate-of-strain components for the element at (r,θ,z) at time t; we then replace r, θ, z, t by r', θ', z', t' and use equations (10). In this way, we obtain

$$\left.\begin{aligned} e^{(1)rr}(r',z',t') &= e^{(1)rr}(r,z,t,t') = (\partial u/\partial r)\,e^{i\sigma t'},\\ e^{(1)\theta\theta}(r',z',t') &= e^{(1)\theta\theta}(r,z,t,t') = (u/r^3)\,e^{i\sigma t'},\\ e^{(1)zz}(r',z',t') &= e^{(1)zz}(r,z,t,t') = (\partial w/\partial z)\,e^{i\sigma t'},\\ e^{(1)\theta z}(r',z',t') &= e^{(1)\theta z}(r,z,t,t') = \tfrac{1}{2}r^{-1}(\partial v/\partial z)\,e^{i\sigma t'},\\ e^{(1)rz}(r',z',t') &= e^{(1)rz}(r,z,t,t') = \tfrac{1}{2}[\partial u/\partial z+\partial w/\partial r]\,e^{i\sigma t'},\\ e^{(1)r\theta}(r',z',t') &= e^{(1)r\theta}(r,z,t,t') = \frac{1}{2}\left[\frac{d}{dr}\left(\frac{V}{r}\right)+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)e^{i\sigma t'}-\frac{u}{i\sigma}\frac{d^2}{dr^2}\left[\frac{V}{r}\right](e^{i\sigma t}-e^{i\sigma t'})\right]. \end{aligned}\right\} \tag{11}$$

† Shield & Green (1963) have recently critized *linear* stability analyses. However, we feel that a linear analysis is justified in the problem under consideration (cf. Chandrasekhar 1961; Taylor 1923).

Equations (10) and (11) can now be used to determine the physical components of the partial-stress tensor. After much reduction, and use of (5)–(7), we obtain

$$\left.\begin{aligned}
p'_{(rr)} &= 2\beta_0(\partial u/\partial r)\, e^{i\sigma t},\\
p'_{(rz)} &= \beta_0(\partial w/\partial r + \partial u/\partial z)\, e^{i\sigma t},\\
p'_{(zz)} &= 2\beta_0(\partial w/\partial z)\, e^{i\sigma t},\\
p'_{(\theta z)} &= \beta_0 \frac{\partial v}{\partial z} e^{i\sigma t} - \frac{2D}{r^2}\delta_0 \frac{\partial w}{\partial r} e^{i\sigma t} - \frac{2D}{r^2}\gamma_0 \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) e^{i\sigma t},\\
p'_{(r\theta)} &= -\frac{2D}{r^2}\eta_0 + \beta_0 r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) e^{i\sigma t} - \frac{2D}{r^2}\delta_0 \left(\frac{\partial u}{\partial r} + \frac{3u}{r}\right) e^{i\sigma t} - \frac{4D}{r^2}\gamma_0 \frac{\partial u}{\partial r} e^{i\sigma t},\\
p'_{(\theta\theta)} &= \frac{8K_0 D^2}{r^4} + 2\beta_0 \frac{u}{r} e^{i\sigma t} + \frac{48D^2}{r^4}\frac{\kappa_0}{i\sigma}\frac{u}{r} e^{i\sigma t} + \frac{16D^2}{r^4} N_0 \frac{\partial u}{\partial r} e^{i\sigma t}\\
&\quad - \frac{4D}{r}\delta_0 \left[\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \frac{2D}{r^3}\frac{1}{i\sigma}\frac{\partial u}{\partial r} + \frac{6D}{r^3}\frac{1}{i\sigma}\frac{u}{r}\right] e^{i\sigma t}\\
&\quad - \frac{4D}{r}\gamma_0 \left[\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{2D}{r^3}\frac{1}{i\sigma}\frac{\partial u}{\partial r} + \frac{6D}{r^3}\frac{1}{i\sigma}\frac{u}{r}\right] e^{i\sigma t},
\end{aligned}\right\} \quad (12)$$

where

$$\left.\begin{aligned}
\eta_0 &= \int_0^\infty N(\tau)\, d\tau, \quad \kappa_0 = \int_0^\infty \tau N(\tau)\, d\tau, \quad \beta_0 = \int_0^\infty \frac{N(\tau)\, d\tau}{1+i\sigma\tau},\\
\delta_0 &= \int_0^\infty \frac{\tau N(\tau)\, d\tau}{1+i\sigma\tau}, \quad \gamma_0 = \int_0^\infty \frac{\tau N(\tau)\, d\tau}{[1+i\sigma\tau]^2}, \quad N_0 = \int_0^\infty \frac{\tau^2 N(\tau)\, d\tau}{[1+i\sigma\tau]^3}.
\end{aligned}\right\} \quad (13)$$

It is convenient at this stage to make the usual assumption that the annular gap between the cylinders is small compared with the radii of the cylinders (cf. Taylor 1923; Chandrasekhar 1954; Thomas & Walters 1964*a*, *b*). Substituting $r = r_1 + dx$, where $d = r_2 - r_1$, and assuming that d/r is small, we have

$$C \doteqdot r_1 \alpha\Omega_1/2d, \quad D \doteqdot -r_1^3 \alpha\Omega_1/2d, \quad C + D/r^2 = \Omega_1(1+\alpha x),$$

where $\alpha = (\Omega_2/\Omega_1) - 1$. On making this approximation in the equations obtained by substituting the stress distribution (12) into the stress equations of motion, we obtain

$$\begin{aligned}
i\sigma\rho u - 2v\Omega_1\rho(1+\alpha x) &= -\frac{1}{d}\frac{\partial p^*}{\partial x} + \beta_0\left[\frac{1}{d^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right] - \frac{12K_0}{i\sigma}\frac{\alpha^2}{d^2}\Omega_1^2 u\\
&\quad - \frac{4N_0 r_1 \alpha^2\Omega_1^2}{d^3}\frac{\partial u}{\partial x} - \frac{2\delta_0\alpha\Omega_1}{d^2}\left[\frac{\partial v}{\partial x} + \frac{r_1\alpha\Omega_1}{i\sigma d}\frac{\partial u}{\partial x} - \frac{3\alpha\Omega_1 u}{i\sigma}\right]\\
&\quad - \frac{2\gamma_0\alpha\Omega_1}{d^2}\left[\frac{\partial v}{\partial x} - \frac{r_1\alpha\Omega_1}{i\sigma d}\frac{\partial u}{\partial x} - \frac{3\alpha\Omega_1 u}{i\sigma}\right],
\end{aligned} \quad (14)$$

$$i\sigma\rho v + u\rho r_1 \frac{\alpha\Omega_1}{d} = \beta_0\left[\frac{1}{d^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\gamma_0 r_1 \alpha\Omega_1}{d}\left[\frac{1}{d^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right], \quad (15)$$

$$i\sigma\rho w = -\frac{\partial p^*}{\partial z} + \beta_0\left[\frac{1}{d^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right], \quad (16)$$

where $p^* e^{i\sigma t}$ is the additional pressure due to the disturbance. The associated equation of continuity is

$$\frac{1}{d}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (17)$$

We now make the further assumption that the disturbance velocities are spatially periodic in the z direction. It is then possible to non-dimensionalize the equations of motion by means of the following substitutions:

$$\left.\begin{aligned}
u &= \epsilon R r_1 \Omega_1 \chi(x) \sin \lambda z, \\
v &= \epsilon r_1 \Omega_1 v_1(x) \sin \lambda z, \\
w &= R r_1 \Omega_1 (d\chi/dx) \cos \lambda z, \\
p^* &= \eta_0 r_1 \Omega_1 (\epsilon R/d)\, p_1(x) \sin \lambda z, \quad R = d^2\Omega_1 \rho/\eta_0, \quad \epsilon = \lambda d, \\
T &= -2\alpha r_1 R^2/d, \quad \bar{\sigma} = d^2\rho\sigma/\eta_0, \quad \bar{\beta}_0 = \beta_0/\eta_0, \quad \bar{\kappa}_0 = \kappa_0/\rho d^2, \\
\bar{\gamma}_0 &= \gamma_0/\rho d^2, \quad \bar{\delta}_0 = \delta_0/\rho d^2, \quad \bar{N}_0 = \eta_0 N_0/\rho^2 d^4.
\end{aligned}\right\} \tag{18}$$

Equations (14)–(16) become

$$\begin{aligned}
i\bar{\sigma}\chi - 2v_1(1+\alpha x) = -dp_1/dx + \bar{\beta}_0 \nabla_1^2 \chi - 2\alpha (dv_1/dx)[\bar{\delta}_0 + \bar{\gamma}_0] \\
+ \alpha T[2\bar{N}_0 + (\bar{\delta}_0/i\bar{\sigma}) - (\bar{\gamma}_0/i\bar{\sigma})]\, d\chi/dx,
\end{aligned} \tag{19}$$

$$i\bar{\sigma} v_1 - \tfrac{1}{2}T\chi = \bar{\beta}_0 \nabla_1^2 v_1 - \tfrac{1}{2}T\bar{\gamma}_0 \nabla_1^2 \chi, \tag{20}$$

$$i\bar{\sigma}\, d\chi/dx = -\epsilon^2 p_1 + \bar{\beta}_0[d^3\chi/dx^3 - \epsilon^2 d\chi/dx], \tag{21}$$

where $\nabla_1^2 \equiv (d^2/dx^2) - \epsilon^2$. The equation of continuity (17) is satisfied identically.

Eliminating p_1 between equations (19) and (21), and writing $v_1 = v_0/2\epsilon^2$ we obtain

$$\begin{aligned}
i\bar{\sigma}\nabla_1^2\chi + v_0(1+\alpha x) = \bar{\beta}_0 \nabla_1^4 \chi + \alpha (dv_0/dx)[\bar{\delta}_0 + \bar{\gamma}_0] \\
- \alpha\epsilon^2 T[2\bar{N}_0 + (\bar{\delta}_0/i\bar{\sigma}) - (\bar{\gamma}_0/i\bar{\sigma})]\, d\chi/dx,
\end{aligned} \tag{22}$$

$$-i\bar{\sigma} v_0 + \bar{\beta}_0 \nabla_1^2 v_0 = -T\epsilon^2[\chi - \bar{\gamma}_0 \nabla_1^2 \chi]. \tag{23}$$

The boundary conditions to be associated with equations (22) and (23) are

$$v_0 = \chi = d\chi/dx = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = 1. \tag{24}$$

To proceed further, it is necessary to specify the distribution function $N(\tau)$ appearing in equations (3) and (13). In part 2 the discussion of the results was confined to a consideration of one particular elastico-viscous model—the Maxwell liquid with one relaxation time at $\tau = \lambda$ (say). For this liquid

$$N(\tau) = \eta_0 \delta(\tau - \lambda). \tag{25}$$

In order to extend the work contained in part 2 and answer some of the questions posed in that paper, we propose in the following to confine attention to the Maxwell model.

Substituting (25) into (13) and writing $k = \eta_0 \lambda/\rho d^2$, we obtain

$$\begin{aligned}
&\bar{\beta}_0 = 1/(1+i\bar{\sigma}k), \quad \bar{\kappa}_0 = k, \\
&\bar{\delta}_0 = k/[1+i\bar{\sigma}k], \quad \bar{\gamma}_0 = k/[1+i\bar{\sigma}k]^2, \quad \bar{N}_0 = k^2/[1+i\bar{\sigma}k]^3,
\end{aligned} \tag{26}$$

and equations (22) and (23) reduce to

$$i\bar{\sigma}\nabla_1^2\chi + v_0(1+\alpha x) = \frac{1}{1+i\bar{\sigma}k}\nabla_1^4\chi + \frac{\alpha k(2+i\bar{\sigma}k)}{(1+i\bar{\sigma}k)^2}\frac{dv_0}{dx} - \frac{\alpha\epsilon^2 T k^2(3+i\bar{\sigma}k)}{(1+i\bar{\sigma}k)^3}\frac{d\chi}{dx}, \tag{27}$$

$$-i\bar{\sigma} v_0 + \frac{1}{1+i\bar{\sigma}k}\nabla_1^2 v_0 = -T\epsilon^2\left[\chi - \frac{k}{(1+i\bar{\sigma}k)^2}\nabla_1^2\chi\right]. \tag{28}$$

Equations (27) and (28) subject to (24) determine a characteristic-value problem for the Taylor number T as a function of the wave-number ϵ, $\bar{\sigma}$, α and k. When $\bar{\sigma} = 0$, these equations reduce to those given by Thomas & Walters (1964*a*, *b*) for the case of 'stationary' stability. In their work, Thomas and Walters found that for sufficiently large k, no real values of T exist for any value of ϵ. In the present paper, it is shown that it is possible to choose a non-zero $\bar{\sigma}$ for which the Taylor number *is* real.

When $k = 0$, equations (27) and (28) subject to (24) are those governing 'overstability' in Newtonian fluids.

3. The solution of the equations

The solution of the characteristic-value problem presented by equations (24), (27) and (28) has been obtained numerically using an initial-value technique. We first multiply equations (27) and (28) by $1+i\bar{\sigma}k$ and separate the resulting equations into their real and imaginary parts. Writing

$$\chi = \chi_R + i\chi_I, \quad v_0 = v_R + iv_I, \quad T = T_R + iT_I, \tag{29}$$

we obtain

$$\begin{aligned} -\bar{\sigma}(\nabla_1^2\chi_I + \bar{\sigma}k\nabla_1^2\chi_R) + (1+\alpha x)(v_R - \bar{\sigma}kv_I) \\ = \nabla_1^4\chi_R + \frac{\alpha k}{1+\bar{\sigma}^2k^2}\left\{(2+\bar{\sigma}^2k^2)\frac{dv_R}{dx} + \bar{\sigma}k\frac{dv_I}{dx}\right\} \\ - \frac{\alpha\epsilon^2k^2}{(1+\bar{\sigma}^2k^2)^2}\left[\left\{(3-\bar{\sigma}^2k^2)\frac{d\chi_R}{dx} + \bar{\sigma}k(5+\bar{\sigma}^2k^2)\frac{d\chi_I}{dx}\right\}T_R \right. \\ \left. -\left\{(3-\bar{\sigma}^2k^2)\frac{d\chi_I}{dx} - \bar{\sigma}k(5+\bar{\sigma}^2k^2)\frac{d\chi_R}{dx}\right\}T_I\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \bar{\sigma}(\nabla_1^2\chi_R - \bar{\sigma}k\nabla_1^2\chi_I) + (1+\alpha x)(v_I + \bar{\sigma}kv_R) \\ = \nabla_1^4\chi_I + \frac{\alpha k}{1+\bar{\sigma}^2k^2}\left\{(2+\bar{\sigma}^2k^2)\frac{dv_I}{dx} - \bar{\sigma}k\frac{dv_R}{dx}\right\} \\ - \frac{\alpha\epsilon^2k^2}{(1+\bar{\sigma}^2k^2)^2}\left[\left\{(3-\bar{\sigma}^2k^2)\frac{d\chi_I}{dx} - \bar{\sigma}k(5+\bar{\sigma}^2k^2)\frac{d\chi_R}{dx}\right\}T_R \right. \\ \left. +\left\{(3-\bar{\sigma}^2k^2)\frac{d\chi_R}{dx} + \bar{\sigma}k(5+\bar{\sigma}^2k^2)\frac{d\chi_I}{dx}\right\}T_I\right], \end{aligned} \tag{31}$$

$$\begin{aligned} \bar{\sigma}(v_I + \bar{\sigma}kv_R) + \nabla_1^2 v_R = -\epsilon^2\left[\left\{\chi_R - \bar{\sigma}k\chi_I - \frac{k}{1+\bar{\sigma}^2k^2}(\nabla_1^2\chi_R + \bar{\sigma}k\nabla_1^2\chi_I)\right\}T_R \right. \\ \left. -\left\{\chi_I + \bar{\sigma}k\chi_R - \frac{k}{1+\bar{\sigma}^2k^2}(\nabla_1^2\chi_I - \bar{\sigma}k\nabla_1^2\chi_R)\right\}T_I\right], \end{aligned} \tag{32}$$

$$\begin{aligned} -\bar{\sigma}(v_R - \bar{\sigma}kv_I) + \nabla_1^2 v_I = -\epsilon^2\left[\left\{\chi_I + \bar{\sigma}k\chi_R - \frac{k}{1+\bar{\sigma}^2k^2}(\nabla_1^2\chi_I - \bar{\sigma}k\nabla_1^2\chi_R)\right\}T_R \right. \\ \left. -\left\{\chi_R - \bar{\sigma}k\chi_I - \frac{k}{1+\bar{\sigma}^2k^2}(\nabla_1^2\chi_R + \bar{\sigma}k\nabla_1^2\chi_I)\right\}T_I\right]. \end{aligned} \tag{33}$$

The appropriate boundary conditions are

$$\chi_R = d\chi_R/dx = v_R = \chi_I = d\chi_I/dx = v_I = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = 1. \tag{34}$$

Since equations (27) and (28) are linear and homogeneous and the boundary conditions (24) are homogeneous, there is an arbitrary amplitude in the solution. To remove this, we impose the further condition

$$d^2\chi/dx^2 = 1+i \quad \text{at} \quad x = 0,$$

i.e.
$$d^2\chi_R/dx^2 = 1, \quad d^2\chi_I/dx^2 = 1 \quad \text{at} \quad x = 0. \tag{35}$$

This normalizing procedure ensures that derivatives of χ and v_0 at the origin are uniquely defined for any given characteristic value of T. For convenience, we write

$$\left.\frac{d^3\chi}{dx^3}\right|_{x=0} = \xi + i\zeta \text{ and } \left.\frac{dv_0}{dx}\right|_{x=0} = \beta + i\gamma, \tag{36}$$

and as a first approximation to one characteristic solution we choose

$$\xi = \xi^{(1)}, \quad \zeta = \zeta^{(1)}, \quad \beta = \beta^{(1)}, \quad \gamma = \gamma^{(1)}, \quad T_R = T_R^{(1)}, \quad T_I = T_I^{(1)}. \tag{37}$$

An initial-value problem is now defined by equations (30)–(33) with boundary conditions

$$\left.\begin{aligned} \chi_R = d\chi_R/dx = 0, \quad d^2\chi_R/dx^2 = 1, \quad d^3\chi_R/dx^3 = \xi^{(1)}, \quad v_R = 0, \quad dv_R/dx = \beta^{(1)}, \\ \chi_I = d\chi_I/dx = 0, \quad d^2\chi_I/dx^2 = 1, \quad d^3\chi_I/dx^3 = \zeta^{(1)}, \quad v_I = 0, \quad dv_I/dx = \gamma^{(1)}, \end{aligned}\right\} \tag{38}$$

at $x = 0$. We integrate this system of equations numerically using a standard Runge–Kutta process as far as $x = 1$. The numerical solution so generated will be a function of ξ, ζ, β, γ, T_R and T_I, which we regard as continuous variables.

We require to satisfy conditions (34) at $x = 1$. It is therefore necessary to determine what changes are produced at $x = 1$ in χ_R, $d\chi_R/dx$, v_R, χ_I, $d\chi_I/dx$ and v_I by small changes $\delta\xi$, $\delta\zeta$, $\delta\beta$, $\delta\gamma$, δT_R and δT_I in ξ, ζ, β, γ, T_R and T_I, respectively. The necessary first-order corrections to the assumed values of ξ, ζ, β, γ, T_R and T_I are obtained using Newton's Rule. Thus, if the required characteristic values are denoted by $\xi^{(0)}$, $\zeta^{(0)}$, $\beta^{(0)}$, $\gamma^{(0)}$, $T_R^{(0)}$ and $T_I^{(0)}$, and if further

$$\xi^{(0)} = \xi^{(1)} + \delta\xi, \text{ etc.}, \tag{39}$$

we obtain, to first order

$$\begin{bmatrix} \frac{\partial\chi_R}{\partial\xi} & \frac{\partial\chi_R}{\partial\zeta} & \frac{\partial\chi_R}{\partial\beta} & \frac{\partial\chi_R}{\partial\gamma} & \frac{\partial\chi_R}{\partial T_R} & \frac{\partial\chi_R}{\partial T_I} \\ \frac{\partial\chi_I}{\partial\xi} & \frac{\partial\chi_I}{\partial\zeta} & \frac{\partial\chi_I}{\partial\beta} & \frac{\partial\chi_I}{\partial\gamma} & \frac{\partial\chi_I}{\partial T_R} & \frac{\partial\chi_I}{\partial T_I} \\ \frac{\partial}{\partial\xi}\left(\frac{d\chi_R}{dx}\right) & \frac{\partial}{\partial\zeta}\left(\frac{d\chi_R}{dx}\right) & \frac{\partial}{\partial\beta}\left(\frac{d\chi_R}{dx}\right) & \frac{\partial}{\partial\gamma}\left(\frac{d\chi_R}{dx}\right) & \frac{\partial}{\partial T_R}\left(\frac{d\chi_R}{dx}\right) & \frac{\partial}{\partial T_I}\left(\frac{d\chi_R}{dx}\right) \\ \frac{\partial}{\partial\xi}\left(\frac{d\chi_I}{dx}\right) & \frac{\partial}{\partial\zeta}\left(\frac{d\chi_I}{dx}\right) & \frac{\partial}{\partial\beta}\left(\frac{d\chi_I}{dx}\right) & \frac{\partial}{\partial\gamma}\left(\frac{d\chi_I}{dx}\right) & \frac{\partial}{\partial T_R}\left(\frac{d\chi_I}{dx}\right) & \frac{\partial}{\partial T_I}\left(\frac{d\chi_I}{dx}\right) \\ \frac{\partial v_R}{\partial\xi} & \frac{\partial v_R}{\partial\zeta} & \frac{\partial v_R}{\partial\beta} & \frac{\partial v_R}{\partial\gamma} & \frac{\partial v_R}{\partial T_R} & \frac{\partial v_R}{\partial T_I} \\ \frac{\partial v_I}{\partial\xi} & \frac{\partial v_I}{\partial\zeta} & \frac{\partial v_I}{\partial\beta} & \frac{\partial v_I}{\partial\gamma} & \frac{\partial v_I}{\partial T_R} & \frac{\partial v_I}{\partial T_I} \end{bmatrix} \begin{bmatrix} \delta\xi \\ \delta\zeta \\ \delta\beta \\ \delta\gamma \\ \delta T_R \\ \delta T_I \end{bmatrix} = - \begin{bmatrix} \chi_R \\ \chi_I \\ \frac{d\chi_R}{dx} \\ \frac{d\chi_I}{dx} \\ v_R \\ v_I \end{bmatrix}, \tag{40}$$

in which all functions are evaluated at $x = 1$ with $\xi = \xi^{(1)}, \zeta = \zeta^{(1)}, \beta = \beta^{(1)}, \gamma = \gamma^{(1)}$, $T_R = T_R^{(1)}$ and $T_I = T_I^{(1)}$. We obtain differential equations for the functions $\partial\chi_R/\partial\xi$ etc. by differentiating (30)–(33) with respect to ξ, ζ, β, γ, T_R and T_I in turn. The appropriate boundary conditions at $x = 0$ to be imposed on these equations are obtained by likewise differentiating the conditions (38).

The initial-value problem now consists of a set of simultaneous differential equations of order 84 subject to 84 conditions at $x = 0$. We integrate this set of equations numerically as far as $x = 1$, there obtaining the corrections to the chosen values of ξ, ζ, β, γ, T_R and T_I from (40). The whole process is then repeated with $\xi^{(2)} = \xi^{(1)} + \delta\xi$, etc., and so on until a satisfactory convergence has been achieved.

Pilot calculations were performed using an IBM 1620 computer with 20K store. The amount of computation involved was however quite considerable and each characteristic value required one hour of computer time for its evaluation. Most of the subsequent results were obtained using the NIRNS Atlas computer where each characteristic value could be found in approximately 3 sec.

4. Numerical results

(i) *The viscous case*

Chandrasekhar (1961) has considered the overstability of viscous fluids and has concluded from his analysis that overstability is not possible for values of α greater than -1. No consideration was given to values of α less than -1 and it was pointed out by Chandrasekhar that it would be particularly worthwhile to explore the case $\alpha = -2$.

Using the numerical method outlined in the last section, it has been possible to consider values of α less than -1. Particular attention has been paid to the case $\alpha = -2$. Figures 1 and 2 contain graphs of T_R and T_I for $\alpha = -2$ and various values of ϵ and $\bar{\sigma}$. It will be observed that no real $\bar{\sigma}$ exists (other than $\bar{\sigma} = 0$) for which $T_I = 0$, so that overstability is not possible for $\alpha = -2$.

Other values of α were considered in less detail. From this work, it was concluded that there was no likelihood of overstability down to $\alpha = -3$ at least. No values lower than -3 were considered.

(ii) *The elastico-viscous case*

On account of the excessive computation involved in considering a series of values of α, it was decided to confine attention to $\alpha = -0{\cdot}5$ for illustration purposes. The results contained in part 2 were first checked using the numerical method outlined in § 3. This involved setting $\bar{\sigma} = 0$ in equations (27) and (28). Over the range where the approximate method used in part 2 was assumed to be accurate, there was excellent agreement between the two methods. It was also confirmed (cf. part 2) that for $\alpha = -0{\cdot}5$, $k = 0{\cdot}8$, $\bar{\sigma} = 0$, no real values of T exist (i.e. $T_I \neq 0$) for any value of ϵ (see figure 3). In part 2, this and other evidence was taken to indicate that the principle of exchange of stabilities does not hold for highly elastic liquids and overstability is possible. In the present paper, this conclusion is confirmed.

Figures 4 and 5 contain graphs of T_R and T_I against $\bar{\sigma}$ for $\epsilon = 3{\cdot}0$ and various values of k. It will be observed that for sufficiently large k, there exists at least one value of $\bar{\sigma}$ (other than $\bar{\sigma} = 0$) for which T_I is zero. When more than one such

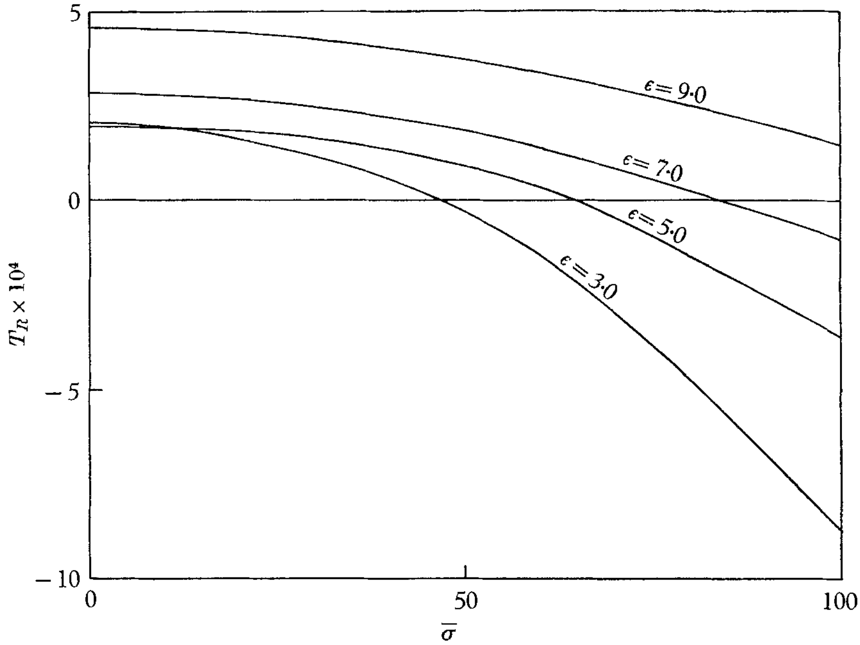

FIGURE 1. Graphs of T_R against $\bar{\sigma}$ for various values of ϵ: $\alpha = -2{\cdot}0$, $k = 0$.

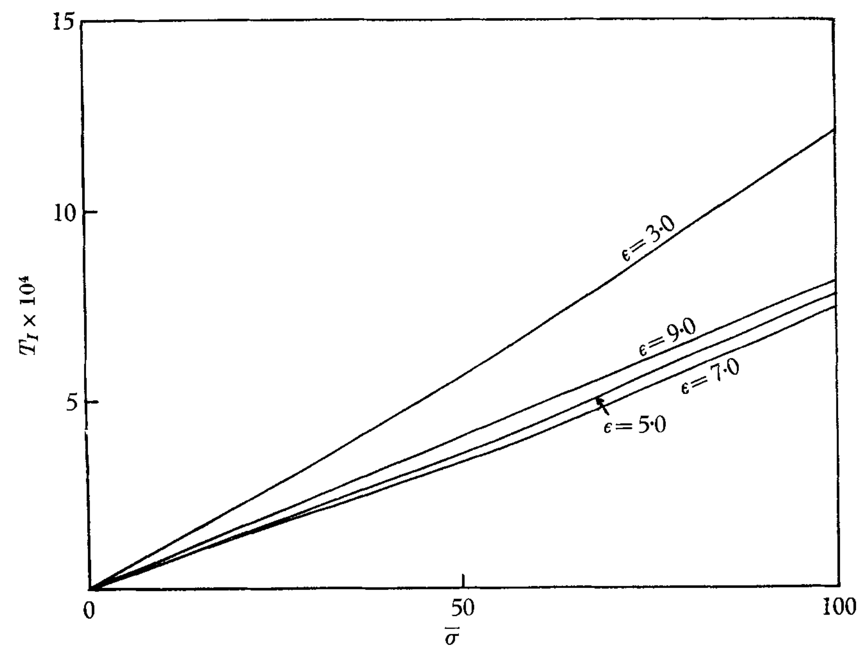

FIGURE 2. Graphs of T_I against $\bar{\sigma}$ for various values of ϵ: $\alpha = -2{\cdot}0$, $k = 0$.

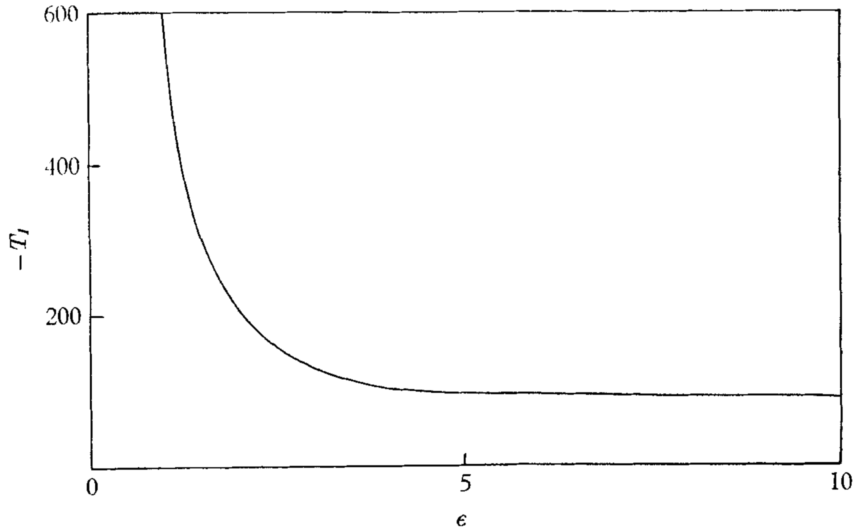

FIGURE 3. Graph of T_I against ϵ for $\alpha = -0{\cdot}5$, $k = 0{\cdot}8$, $\bar{\sigma} = 0$.

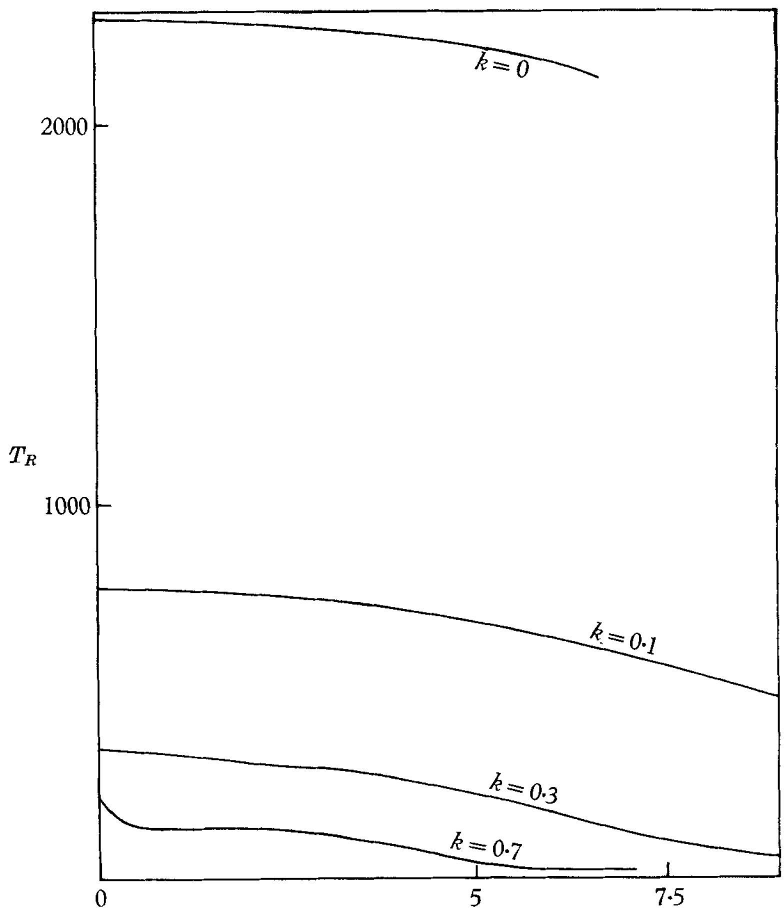

FIGURE 4. Graphs of T_R against $\bar{\sigma}$ for various values of k: $\alpha = -0{\cdot}5$, $\epsilon = 3{\cdot}0$.

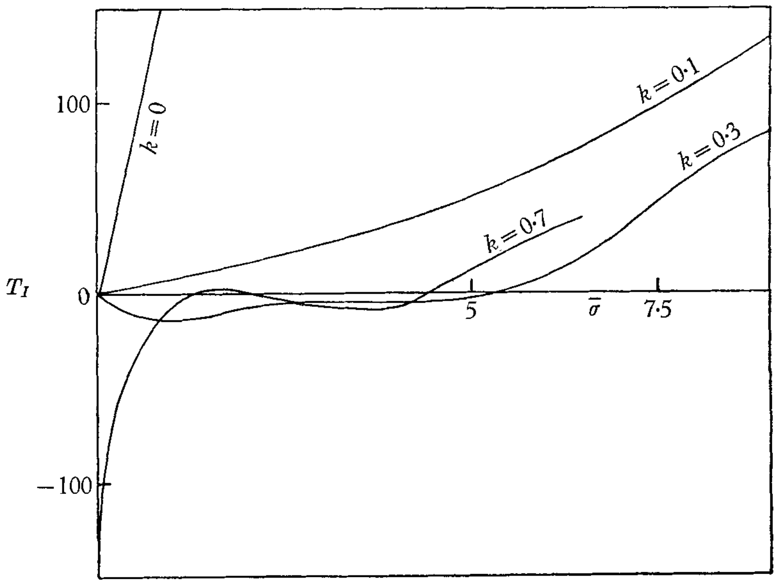

FIGURE 5. Graphs of T_I against $\bar{\sigma}$ for various values of k: $\alpha = -0{\cdot}5$, $\epsilon = 3{\cdot}0$.

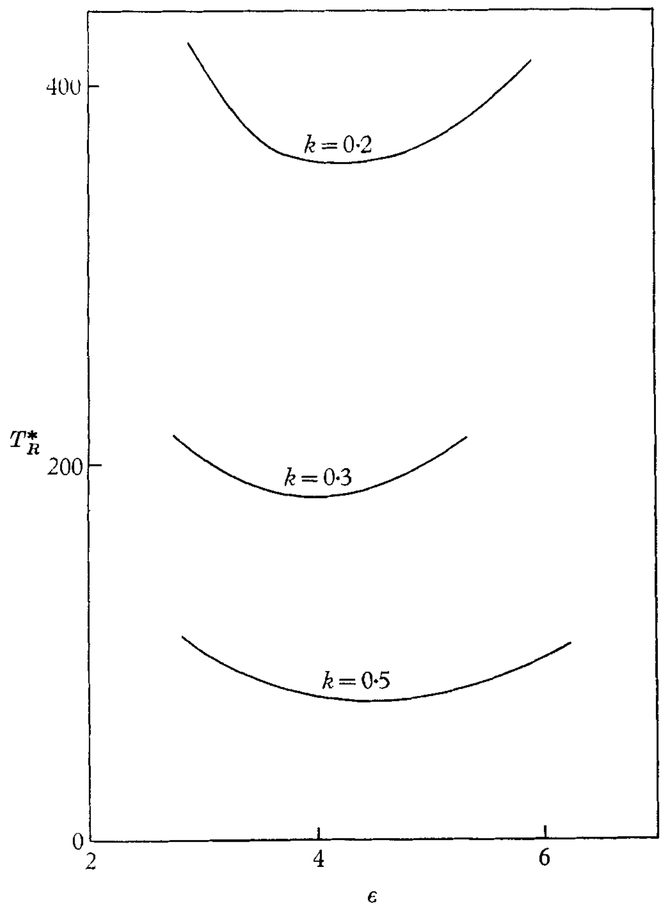

FIGURE 6. Graphs of T_R^* against ϵ for various values of k: $\alpha = -0{\cdot}5$.

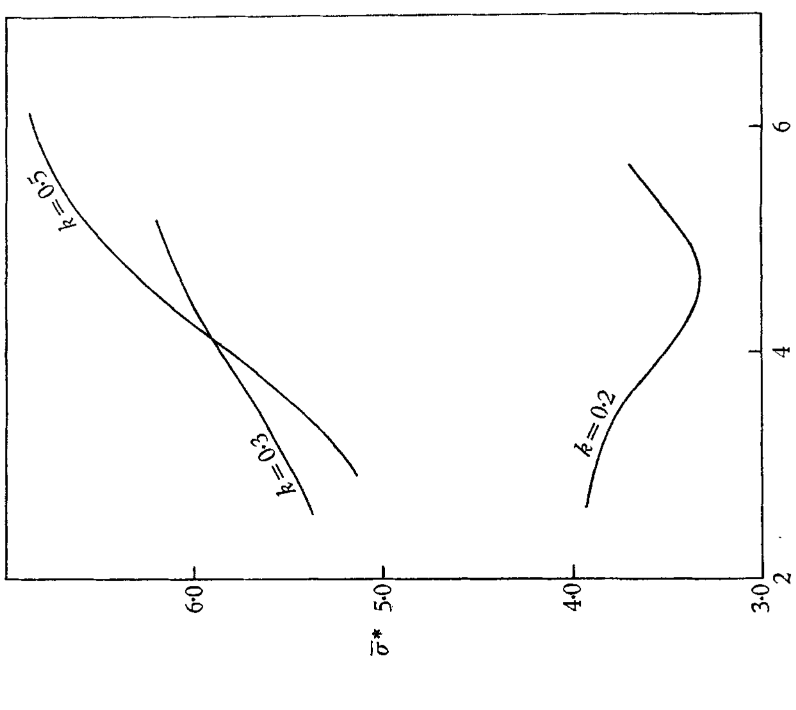

FIGURE 7. Graphs of $\bar{\sigma}^*$ against ϵ for various values of k: $\alpha = -0{\cdot}5$.

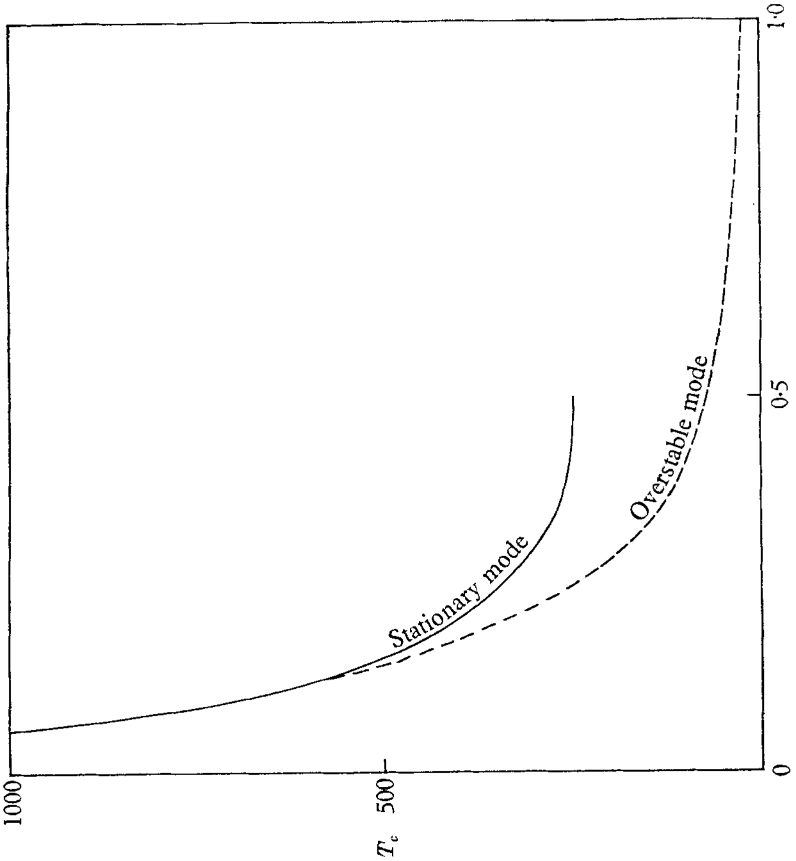

FIGURE 8. Graph of T_c against k: $\alpha = -0{\cdot}5$.

value exists, the particular value of interest is the one associated with the lowest value of T_R. This value of $\bar{\sigma}$ will be denoted by $\bar{\sigma}^*$ and the corresponding value of T_R by T_R^*.

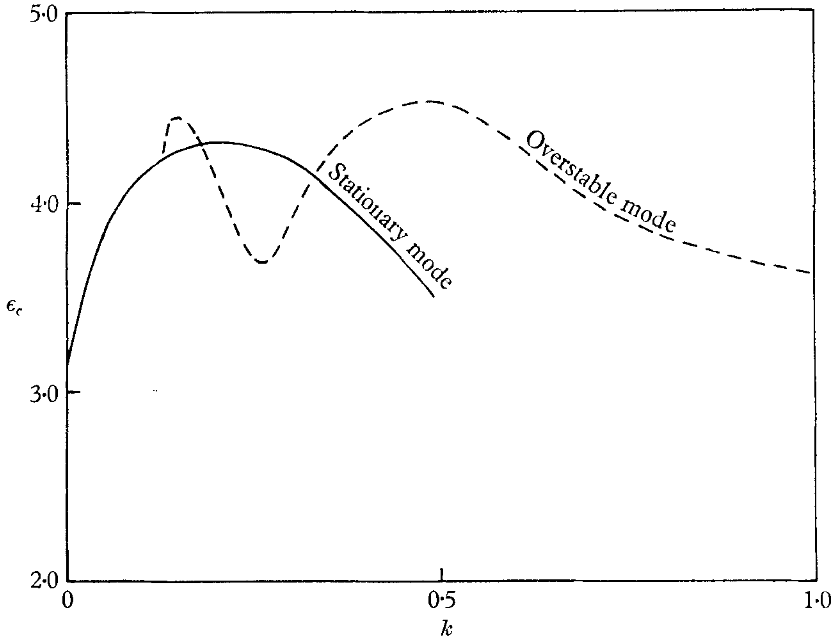

FIGURE 9. Graph of ϵ_c against k: $\alpha = -0{\cdot}5$.

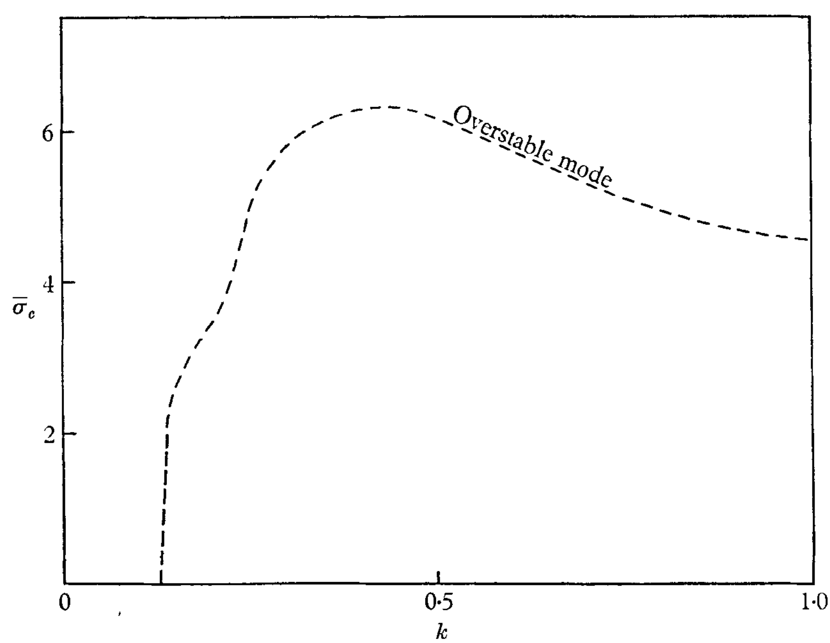

FIGURE 10. Graph of $\bar{\sigma}_c$ against k: $\alpha = -0{\cdot}5$.

Figures 6 and 7 contain graphs of T_R^* and $\overline{\sigma}^*$ against ϵ for various values of k. The critical Taylor number (T_c) at which the laminar flow pattern breaks down can be determined from figure 6 by calculating the minimum T_R^* for varying ϵ. As in the case of the 'stationary' mode, the curves have a well-defined minimum and no difficulty is encountered in determining T_c and the corresponding value of ϵ (ϵ_c). The critical value of $\overline{\sigma}$ ($\overline{\sigma}_c$) can be obtained from figure 7 by reading off the value of $\overline{\sigma}$ corresponding to ϵ_c.

Having referred to the procedure adopted in determining T_c, ϵ_c and $\overline{\sigma}_c$, it is now possible to summarize the major results of the present investigation by means of three figures. Figure 8 contains a graph of T_c against k and figures 9 and 10 contain the corresponding graphs for ϵ_c and $\overline{\sigma}_c$. Also contained in figures 8 and 9 are the 'stationary' mode curves computed in part 2.

For values of k up to (approximately) $k = 0{\cdot}13$, the only T_c which exists corresponds to a stationary stability mode. In this region, the presence of elasticity in the liquid is a significant destablizing agent; the critical Taylor number falling from 2275 at $k = 0$ to 562 at $k = 0{\cdot}13$. For higher values of k, a T_c exists for both a stationary and an overstable mode. Since the critical Taylor number associated with the overstable mode is always lower than that associated with the stationary mode, it follows that overstability sets in at $k = 0{\cdot}13$ and is present for all higher values of k. The critical Taylor number associated with the overstable mode decreases steadily with increasing k and has reached 26·5 when $k = 1{\cdot}0$, so that highly elastic Maxwell liquids can be very unstable indeed.

The conclusions reached in this series of papers refer to a particular elastico-viscous model. However, the authors are of the opinion that consideration of other (more complicated) models is not likely to lead to results which are any less (or more) spectacular. It is therefore suggested that the next step must be an experimental programme designed to verify that elastico-viscous liquids can have stability characteristics which are grossly different from those for a Newtonian liquid.

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